(2) Attempt any **four** questions out of remaining **six** questions.

1. (a) If $y = \dfrac{8x}{x^3 - 2x^2 - 4x + 8}$ find y_n 5

(b) If $| z^2 - 1 | = | z |^2 + 1$ prove that z lies on imaginary axis where z is a complex number. 5

(c) If $u = \log (x^3 + y^3 - x^2y - xy^2)$ prove that— 5

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\, \partial y} + \frac{\partial^2 u}{\partial y^2} = \frac{-4}{(x+y)^2}.$$

(d) Evaluate $\displaystyle\lim_{x \to a} \left[\frac{1}{2}\left(\sqrt{\frac{a}{x}} + \sqrt{\frac{x}{a}} \right) \right]^{\frac{1}{x-a}}$. 5

2. (a) Prove that $\cos^6 \theta - \sin^6 \theta = \dfrac{1}{16}\,[\cos 6\theta + 15 \cos 2\theta]$. 6

(b) Test the convergence of— 6

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \dots\dots$$

(c) If f, g and h are continuous on [a, b] and differentiable on (a, b) prove that there exist $c \in (a, b)$ such that— 8

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0 .$$

Deduce Cauchy's and Lagranges mean value Theorem from this result.

3. (a) If $x + iy = c \cot (u + iv)$ show that— 6

$$\frac{x}{\sin(2u)} = \frac{-y}{\sinh(2v)} = \frac{c}{\cosh(2v) - \cos(2u)}.$$

(b) If $v = \log \sin \left[\dfrac{\pi\left(2x^2 + y^2 + xz\right)^{\frac{1}{2}}}{2\left(x^2 + xy + 2yz + z^2\right)^{1/3}} \right]$ show that— 6

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + z\frac{\partial v}{\partial z} = \frac{1}{3} e^{-v} \sqrt{1 - e^{2v}}\ \sin^{-1}\left(e^{v}\right).$$

(c) If $y = \dfrac{x}{x^2 + a^2}$ prove that $y_n = \dfrac{(-1)^n\, n!}{a^{n+1}} \sin^{n+1}\theta \cos (n+1)\theta$. 8

4. (a) Considering only principle value, if $(1 + i \tan \alpha)^{1 + i \tan \beta}$ is real prove that its value is $(\sec \alpha)^{\sec^2 \beta}$. 6

(b) Show that $x \operatorname{cosec} x = 1 + \dfrac{x^2}{6} + \dfrac{7}{360} x^4 + \dots\dots$ 6

(c) If $\bar{r} = xi + yj + zk$ and $\bar{a}, \bar{b}$ are constant vectors, prove that 8

$$\bar{a} \cdot \nabla\left(\bar{b} \cdot \nabla\frac{1}{r}\right) = 3\,\frac{(\bar{a} \cdot \bar{r})(\bar{b} \cdot \bar{r})}{r^5} - \frac{\bar{a} \cdot \bar{b}}{r^3}.$$

5. (a) Find the directional derivative of $\phi = x^2 + y^2 + z^2$ in the direction of the line $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ at (1, 2, 3). **6**

(b) If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ **6**

Show that— $6\frac{\partial u}{\partial x} + 4\frac{\partial u}{\partial y} + 3\frac{\partial u}{\partial z} = 0$

(c) Find a point in the plane $x + 2y + 3z = 13$ nearest to the point (1, 1, 1) using the method of Lagrange's multipliers. **8**

6. (a) Show that $\sin^{-1}(x) = x + \frac{x^3}{6} + \frac{3}{40}x^5 + \ldots\ldots\ldots$ **6**

(b) State and prove Euler's theorem for two variables. **6**

(c) Prove that nth root of unity are in geometric progression. Also find sum of nth root of unity. **8**

7. (a) Find $(1{\cdot}04)^{3{\cdot}01}$ by using theory of approximation. **6**

(b) If $y^{1/m} + y^{-1/m} = 2x$ prove that — **6**

$(x^2 - 1)\, y_{n+2} + (2n + 1)\, x y_{n+1} + (n^2 - m^2)\, y_n = 0.$

(c) If $u = \log \tan\left(\pi/4 + \theta/2\right)$ prove that— **8**

(i) $\cos h\,(u) = \sec(\theta)$

(ii) $\sin h\,(u) = \tan\theta$

(iii) $\tan h\,(u) = \sin\theta$

(iv) $\tan h\,(u/2) = \tan(\theta/2)$.